Dynamics of a Threshold Public Goods Game in Ambiguity

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1. Context

1.7 million species have been identified, but this number is believed to be to a great extent higher. Climate change has produced shifts in the distribution and abundance of species (Thomas et al. 2004, Wright and Muller-Landau 2006); species are likely to become extinct and some of them face extinction before even being identified.

Species have a quasi-option value (Arrow and Fisher 1974) made available through their preservation.

Their loss deprives of tools for biomedical research for all untreatable diseases (Chivian and Bernstein 2004).

In order to lower this collective loss, agents can jointly produce public goods if they attain the threshold level of cost to produce the public good.

But all agents have an incentive to free-ride: social dilemma.

Key question: do agents contribute to the public good in ambiguity?
2. Compound probability

- Does the species exist? Lottery $A$
- Will the species survive? Lottery $B$
- Will the species be rapidly useful for medical treatment? Lottery $C$

This is a three-stage lottery $A: (B, \gamma; 0, 1-\gamma)$, $B: (C, p; 0, 1-p)$ and $C: (w, q; 0, 1-q)$. 
Transforming into a reduced compound lottery $\mathbf{D}^3$ with a single stage yields $\mathbf{D}^3$: $(w, \gamma pq; 0, \gamma p(1-q); 0, 1-\gamma)$

If the compound probability axiom holds, \textit{i.e.} $\mathbf{A} \sim \mathbf{D}^3$, the expected salvage of wealth is $\mathbf{D}^3 \equiv \gamma pq(w) = \phi(w)$.

But, an agent who decides to affect the probability of the species’ survival considers their existence and usefulness as granted, thus $(p \mid \gamma q) = p$ and $\gamma pq(w) \equiv p(w)$. 
The probability of failure of the public good depends on agents’ contributions.

\[ \pi = (1 - p)(w - gk^{-1}) \]

where \( w > 0 \) endowment, \( k \in [0,1] \) is the probability of an agent of being at risk and \( gk^{-1} = G/kN > 0 \) is the proportional fair-share of threshold \( G \) provided by \( N \) agents.

According to prevalence proportion, the probability \( k \) of being at risk of a randomly chosen agent from the population is equal to the proportion of the population at risk.

When \( k = 1 \), the whole population is at risk (common disease), so \( gk^{-1} = G/N \).

As \( k \to 0 \), it becomes more and more costly to fund the public good, given the low probability of being at risk (rare disease).
3. Static threshold public goods game

Agents contribute $gk^{-1}$ to attain the threshold. The population $N$ is divided between contributors $n$ and $N - n$ free-riders.

If $N - n = 0$ the threshold is attained and the survival is assured ($p = 1$). In this case, the analysis ends.

If $ng \leq G \Leftrightarrow n \leq N$ the survival becomes ambiguous. In this case, the payoffs of a contributor and a free-rider are

\[
\begin{align*}
\pi_c &= p(w - gk^{-1}) + (1 - p)(w - gk^{-1}) \\
\pi_f &= (1 - p)w
\end{align*}
\]

The probability that a contributor salvages her wealth is $p$ and is $1 - p$ if there is a free-rider. A free-rider faces $1 - p$ given that she endangers the probability of survival.
When the survival of species is null or $p = 0$

\[
\begin{align*}
\pi_c &= w - gk^{-1} \\
\pi_f &= w
\end{align*}
\]

We have $w > w - gk^{-1}$.

**Result 1.** *Free-riding always dominates because it provides a higher payoff.*
When the survival of species is ambiguous or $p \in [0,1]$ two cases arise

If $\pi_c < \pi_f \iff gk^{-1} > pw$.

**Result 2.** *Free-riding dominates because it provides a higher payoff (the proportional fair-share is greater than the expected salvage of wealth).*

If $\pi_c > \pi_f \iff gk^{-1} < pw$.

**Result 3.** *Contributing dominates because it provides a higher payoff (the proportional fair-share is less than the expected salvage of wealth).*
4. Dynamic threshold public goods game

Population dynamics in a replicator equation. Infinite populations of \( x \) contributors and \( y \) free-riders, where \( x + y = 1 \).

The evolution of the system is given by

\[
\begin{align*}
\dot{x} &= x(f_c - \bar{f}) \\
\dot{y} &= y(f_f - \bar{f})
\end{align*}
\]

\( f_c \) and \( f_f \) are expected payoffs of contributors \( x \) and free-riders \( y \).

\( \bar{f} = xf_c + yf_f \) is the average payoff in the population determined by the interactions in randomly formed groups.
$N$ agents are randomly chosen according to the binomial probability function. The probability that there are $n$ contributors among $N - 1$ agents in the population for a given model-contributor is

$$f(n \mid N - 1, x) = \binom{N - 1}{n} x^n y^{N - 1 - n}.$$ 

The average payoffs of a model-contributor and a model-free-rider are

$$\begin{cases}
  f_c &\equiv (1 - p)(w - gk^{-1}) + p(w - gk^{-1})x^{N - 1} \\
  f_f &\equiv (1 - p)w
\end{cases}$$

The dynamic evolution of $F(x) \equiv x(t)$ amounts to

$$\dot{x} = x(1 - x)[p(w - gk^{-1})x^{N - 1} - (1 - p)gk^{-1}].$$
When the survival of species is null or \( p = 0 \)

\[
\dot{x} = -x(1-x)gk^{-1}.
\]

Solving \( \dot{x} = 0 \) gives two fixed points of the replicator dynamics which cancel out \( x(1-x) \): \( x = 0 \) and \( x = 1 \). We study the stability of those steady states by the Lyapunov method. The derivative of \( F(x) \) gives

\[
F'(x) = -gk^{-1} + 2xgk^{-1}.
\]

At \( x = 0 \), \( F'(0) < 0 \): stable equilibrium.

At \( x = 1 \), \( F'(1) > 0 \): unstable equilibrium.

**Proposition 1.** In case of null survival of the species, free-riding is the only steady state.
When the survival of species is ambiguous or $0 < p < 1$

$$\dot{x} = x(1-x)[p(w - gk^{-1})x^{N-1} - (1-p)gk^{-1}].$$

Fixing $\dot{x} = 0$ gives two fixed points: $x = 0$ and $x = 1$. The derivative of $F(x)$ gives

$$F'(x) = (1-2x)[p(w - gk^{-1})x^{N-1} - (1-p)gk^{-1}] + x(1-x)[(N-1)p(w - gk^{-1})x^{N-2}].$$

At $x = 0$, $F'(0) < 0$: stable equilibrium.

At $x = 1$, $F'(1) \leq 0$. If $gk^{-1} < pw$, the equilibrium is stable. If greater, it is unstable.

**Proposition 2.** Contributing dominates when the proportional fair-share is less than the expected salvage of wealth. Otherwise, the model-agent is better off free-riding.
5. Existence and uniqueness of an interior equilibrium

We reduce $x(t)$ to the function $T(x) = f_c - f_f$. The interior equilibrium has to be the root of $T(x)$ in the interval $[0,1]$. We have $T(0) < 0$ and $T(1) > 0$.

If $gk^{-1} > pw$, there is no interior equilibrium. If $gk^{-1} < pw$, $T(1) > 0$. Furthermore, $p \in [0,1]$ verifies $T(0) < 0$. Up to now, the fulfilled conditions are necessary but not sufficient. At last, we have $T'(x) > 0$ which ends the proof. The equilibrium equals

$$x^* = \left[ \frac{gk^{-1} - pgk^{-1}}{pw - pgk^{-1}} \right]^{1/N}.$$

At $x = x^*$, $T'(x^*) > 0$: unstable equilibrium.

**Proposition 3.** There is a unique unstable Nash equilibrium $x^*$, where the trade-off between the proportional fair-share and the expected salvage of wealth determines whether agents end up contributing or free-riding.


6. Rare disease

When the disease is common or $k \rightarrow 1$, the trade-off $gk^{-1} \leq pw$ depends on the agent’s probability of salvage of wealth.

As the disease turns rare or $k \rightarrow 0$, the constraint against contributing increases up to $gk^{-1} \rightarrow \infty$. Since $pw \rightarrow w$, when $p \rightarrow 1$, we have

$$gk^{-1} \gg pw.$$ 

Low $k$ leads to a cost of contribution greater than what the wealth constraint permits.

**Proposition 4.** Low $k$ provokes social free-riding and disinterest in the public good.
7. Conclusive remarks


In case of ambiguous survival, agents end up contributing if their proportional fair-share (cost of the public good) is lower than their expected salvage of wealth (benefit from the public good).

Agents contribute to the public good when losses are ambiguous, when they are at risk of suffering personal losses (contradicts the results by Milinski et al. 2008).

In case of a rare disease, agents end up free-riding. The results invalidate the argument by Olson (1968) and Marwell and Ames (1979) that public goods are provided by groups in which an individual has an interest in the good that is greater than the cost of the good.

However, if the individual at stake is the model-agent, their argument holds water.
8. Experiment

Experimental parameters: $\gamma = q = 1$ and $p \in [0,1]$ that the threshold is reached. $w = 10$, $G = 20$ thus $g = 20/10 = 2$. Subjects play in infinite time in groups of 10.

Subjects are told that 5/10 people will be randomly chosen to be at risk of losing their earnings: $k = 0.5$.

**Risky treatment:** if the threshold is reached, the probability of losing earnings reduces from 1 to 0.5, thus $p = 0.5$.

**Ambiguous treatment:** if the threshold is reached, the probability of losing earnings reduces from 1 to some probability within $[0,1]$ thus $p \in [0,1]$.

Subjects note down their contribution, their subjective belief that the threshold will be attained ($p$), and their subjective belief of being at risk ($k$).

Subjects learn about total contributions, whether they were at risk, the probability with which they were at risk, and their earnings after all subjects contribute.
The model-agent is represented by the experimental average agent.

<table>
<thead>
<tr>
<th>Risky treatment (50 subjects; on average)</th>
<th>Experiment</th>
<th>Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percentage of contributors</td>
<td>58%</td>
<td>50%</td>
</tr>
<tr>
<td>Public good level (sum of contributions)</td>
<td>27.77</td>
<td>20</td>
</tr>
<tr>
<td>Contribution</td>
<td>2.78</td>
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<tr>
<td>Subjective belief the threshold is attained ($p$)</td>
<td>0.64</td>
<td>0.5</td>
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<tr>
<td>Subjective belief of being at risk ($k$)</td>
<td>0.45</td>
<td>0.5</td>
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<tr>
<td>Proportional fair-share</td>
<td>5.2</td>
<td>4</td>
</tr>
<tr>
<td>Expected salvage of wealth</td>
<td>6.5</td>
<td>5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Ambiguity treatment (50 subjects; on average)</th>
<th>Experiment</th>
<th>Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percentage of contributors</td>
<td>55%</td>
<td>50%</td>
</tr>
<tr>
<td>Public good level (sum of contributions)</td>
<td>23.33</td>
<td>20</td>
</tr>
<tr>
<td>Contribution</td>
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<td>2</td>
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<tr>
<td>Subjective belief the threshold is attained ($p$)</td>
<td>0.59</td>
<td>0.5</td>
</tr>
<tr>
<td>Subjective belief of being at risk ($k$)</td>
<td>0.49</td>
<td>0.5</td>
</tr>
<tr>
<td>Proportional fair-share</td>
<td>6.31</td>
<td>4</td>
</tr>
<tr>
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<td>6.35</td>
<td>5</td>
</tr>
</tbody>
</table>
Risky treatment: public good level (sum of contributions)
Ambiguity treatment: public good level (sum of contributions)
Risky treatment: percentage of contributors
**Ambiguity treatment**: percentage of contributors
**Risky treatment**: subjective belief of threshold probability ($p$)
Ambiguity treatment: subjective belief of threshold probability ($p$)
Risky treatment: subjective belief of being at risk ($k$)
**Ambiguity treatment**: subjective belief of being at risk \((k)\)
THANK YOU
FOR YOUR ATTENTION

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